AMC 12/AHSME 2021

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by P_Groudon, franzliszt, MathPirate101, fidgetboss_4000, Archimedes15, samrocksnature, kevinmathz, advancedjus, franchester, Professor-Mom, knightime1010, Welp..., rrusczyk

– A

– February 4, 2021

1 What is the value of

$$2^{1+2+3}-(2^1+2^2+2^3)?$$

(A) 0 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 57

Proposed by **djmathman**

2 Under what conditions is $\sqrt{a^2+b^2}=a+b$ true, where a and b are real numbers?

(A) It is never true. **(B)** It is true if and only if $ab=0$. **(C)** It is true if and only if $a+b\geq 0$. **(D)** It is true if and only if $ab=0$ and $a+b\geq 0$. **(E)** It is always true.

3 The sum of two natural numbers is $17,402$. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

(A) $10,272$ **(B)** $11,700$ **(C)** $13,362$ **(D)** $14,238$ **(E)** $15,426$

4 Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that • all of his happy snakes can add • none of his purple snakes can subtract, and • all of his snakes that can't subtract also can't add

Which of these conclusions can be drawn about Tom's snakes?

(A) Purple snakes can add. **(B)** Purple snakes are happy. **(C)** Snakes that can add are purple. **(D)** Happy snakes are not purple. **(E)** Happy snakes can't subtract.

5 When a student multiplied the number 66 by the repeating decimal,

$$1.\underline{ab}\underline{ab}\ldots = 1.\overline{ab},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $1.\underline{ab}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2- digit integer $\underline{ab}$?

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

6 A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When 4 black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally.

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

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7 What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

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8 A sequence of numbers is defined by $D_0=0, D_1=0, D_2=1$ and $D_n=D_{n-1}+D_{n-3}$ for $n\geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2023})$, where E denotes even and O denotes odd?

(A) (O,E,O) **(B)** (E,E,O) **(C)** (E,O,E) **(D)** (O,O,E) **(E)** (O,O,O)

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9 Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

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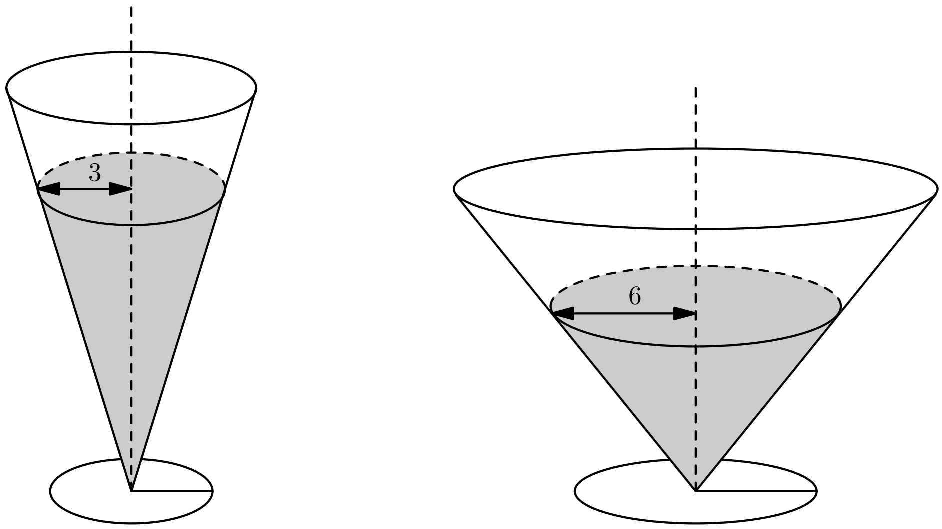
10 Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?

(A) $1:1$ **(B)** $47:43$ **(C)** $2:1$ **(D)** $40:13$ **(E)** $4:1$



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11 A laser is placed at the point (3,5). The laser bean travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7,5)$. What is the total distance the beam will travel along this path?

(A) $2\sqrt{10}$ **(B)** $5\sqrt{2}$ **(C)** $10\sqrt{2}$ **(D)** $15\sqrt{2}$ **(E)** $10\sqrt{5}$

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12 All the roots of polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers. What is the value of B?

(A) -88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

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13 Of the following complex numbers z, which one has the property that z^5 has the greatest real part?

(A) -2 **(B)** $-\sqrt{3}+i$ **(C)** $-\sqrt{2}+\sqrt{2}i$ **(D)** $-1+\sqrt{3}i$ **(E)** $2i$

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14 What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21 **(B)** $100\log_5 3$ **(C)** $200\log_3 5$ **(D)** $2,200$ **(E)** $21,000$

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15 A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that can be selected. What is the remainder when N is divided by 100?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

16 In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

17 Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. GIven that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

18 Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

19 How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0, \pi]$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

20 Suppose that on a parabola with vertex V and a focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

(A) 13 **(B)** $\frac{40}{3}$ **(C)** $\frac{41}{3}$ **(D)** 14 **(E)** $\frac{43}{3}$

Proposed by **djmathman**

21 The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6) = 0$$

may be written in the form $x_k + y_k i$ for $1 \leq k \leq 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$ where m and n are relatively prime positive integers. What is $m + n$? (Recall that the *eccentricity* of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of E and $2c$ is the is the distence between its two foci.)

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Proposed by **djmathman**

22 Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is abc?

(A) $-\frac{3}{49}$ **(B)** $-\frac{1}{28}$ **(C)** $\frac{\sqrt[3]{7}}{64}$ **(D)** $\frac{1}{32}$ **(E)** $\frac{1}{28}$

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23 Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she ”wraps around” and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops ”up”, the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

(A) $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{25}{32}$ **(E)** $\frac{13}{16}$

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24 Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

(A) 110 **(B)** 114 **(C)** 118 **(D)** 122 **(E)** 126

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25 Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the *divisor function*.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

– B

– February 10, 2021

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1 How many integer values satisfy $|x| < 3\pi$?

(A) 9 **(B)** 10 **(C)** 18 **(D)** 19 **(E)** 20

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2 At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 points. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both studets wearing yellow shirts?

(A) 23 **(B)** 32 **(C)** 37 **(D)** 41 **(E)** 64

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3 Suppose

$$2+\frac{1}{1+\frac{1}{2+\frac{2}{3+x}}}=\frac{144}{53}.$$

What is the value of x?

(A) $\frac{3}{4}$ **(B)** $\frac{7}{8}$ **(C)** $\frac{14}{15}$ **(D)** $\frac{37}{38}$ **(E)** $\frac{52}{53}$

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4 Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

(A) 74 **(B)** 75 **(C)** 76 **(D)** 77 **(E)** 78

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5 The point $P(a,b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1,5)$ and then reflected about the line $y=-x$. The image of P after these two transformations is at $(-6,3)$. What is $b-a$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

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6 An inverted cone with base radius 12 cm and height 18 cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24 cm. What is the height in centimeters of the water in the cylinder?

(A) 1.5 **(B)** 3 **(C)** 4 **(D)** 4.5 **(E)** 6

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7 Let $N=34\cdot34\cdot63\cdot270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

(A) $1:16$ **(B)** $1:15$ **(C)** $1:14$ **(D)** $1:8$ **(E)** $1:3$

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8 Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and 34. What is the distance between two adjacent parallel lines?

(A) $5\frac{1}{2}$ **(B)** 6 **(C)** $6\frac{1}{2}$ **(D)** 7 **(E)** $7\frac{1}{2}$

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9 What is the value of

$$\frac{\log_2 80}{\log_{40} 2}-\frac{\log_2 160}{\log_{20} 2}?$$

(A) 0 **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** 2 **(E)** $\log_2 5$

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10 Two distinct numbers are selected from the set $\{1, 2, 3, 4, \ldots, 36, 37\}$ so that the sum of the remaining 35 numbers is the product of these two numbers. What is the difference of these two numbers?

(A) 5 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

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11 Triangle ABC has $AB = 13, BC = 14$ and $AC = 15$. Let P be the point on $\overline{AC}$ such that $PC = 10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

(A) $\frac{42}{5}$ **(B)** $6\sqrt{2}$ **(C)** $\frac{84}{5}$ **(D)** $12\sqrt{2}$ **(E)** 18

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12 Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer is S is *also* removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

(A) 36.2 **(B)** 36.4 **(C)** 36.6 **(D)** 36.8 **(E)** 37

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13 How many values of θ in the interval $0 < \theta \le 2\pi$ satisfy

$$1 - 3\sin\theta + 5\cos 3\theta = 0?$$

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

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14 Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

(A) $24\sqrt{5}$ **(B)** 60 **(C)** $28\sqrt{5}$ **(D)** 66 **(E)** $8\sqrt{70}$

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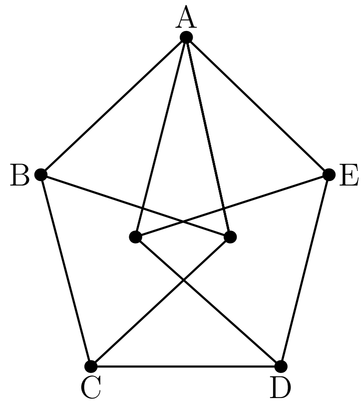
15 The figure below is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?



(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

Proposed by **djmathman**

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16 Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of $a, b,$ and c?

(A) $\frac{1+a+b+c}{c}$ **(B)** $1 + a + b + c$ **(C)** $\frac{1+a+b+c}{c^2}$ **(D)** $\frac{a+b+c}{c^2}$ **(E)** $\frac{1+a+b+c}{a+b+c}$

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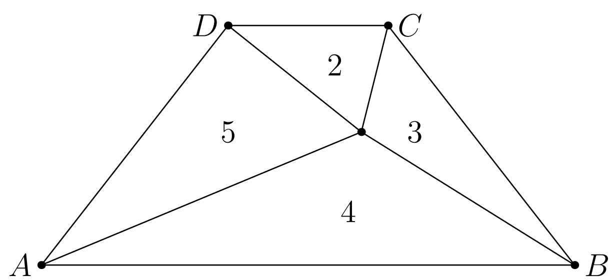
17 Let $ABCD$ be an isoceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4,$ and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



(A) 3 **(B)** $2 + \sqrt{2}$ **(C)** $1 + \sqrt{6}$ **(D)** $2\sqrt{3}$ **(E)** $3\sqrt{2}$

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18 Let z be a complex number satisfying $12|z|^2 = 2|z + 2|^2 + |z^2 + 1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

(A) -2 **(B)** -1 **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 4

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19 Two fair dice, each with at least 6 faces, are rolled. On each face of each die is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum of 7 is $\frac{3}{4}$ of the probability of rolling a sum of 10 and the probability of rolling a sum of 12 is $\frac{1}{12}$. What is the least possible number of faces on the two dice combined?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

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20 Let $Q(z)$ and $R(z)$ be the unique polynomials such that

$$z^{2021}+1=(z^2+z+1)Q(z)+R(z)$$

and the degree of R is less than 2. What is $R(z)$?

(A) $-z$ **(B)** -1 **(C)** 2021 **(D)** $z+1$ **(E)** $2z+1$

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21 Let S be the sum of all positive real numbers x for which

$$x^{2^{\sqrt{2}}}=\sqrt{2}^{2^x}.$$

Which of the following statements is true?

(A) $S<\sqrt{2}$ **(B)** $S=\sqrt{2}$ **(C)** $\sqrt{2}<S<2$ **(D)** $2\le S<6$ **(E)** $S\ge 6$

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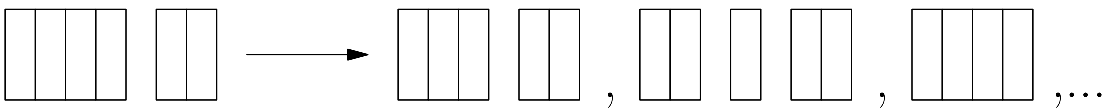
22 Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3,2)$, $(2,1,2)$, (4), $(4,1)$, $(2,2)$, or $(1,1,2)$.



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

(A) $(6,1,1)$ **(B)** $(6,2,1)$ **(C)** $(6,2,2)$ **(D)** $(6,3,1)$ **(E)** $(6,3,2)$

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23 Three balls are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability it is tossed into bin i is 2^{-i} for $i=1,2,3,\ldots$. More than one ball is allowed in each bin. The probability that the balls end up evenly spaced in distinct bins is $\frac{p}{q}$, where p and q are relatively prime positive integers. (For example, the balls are evenly spaced if they are tossed into bins 3, 17, and 10.) What is $p+q$?

(A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

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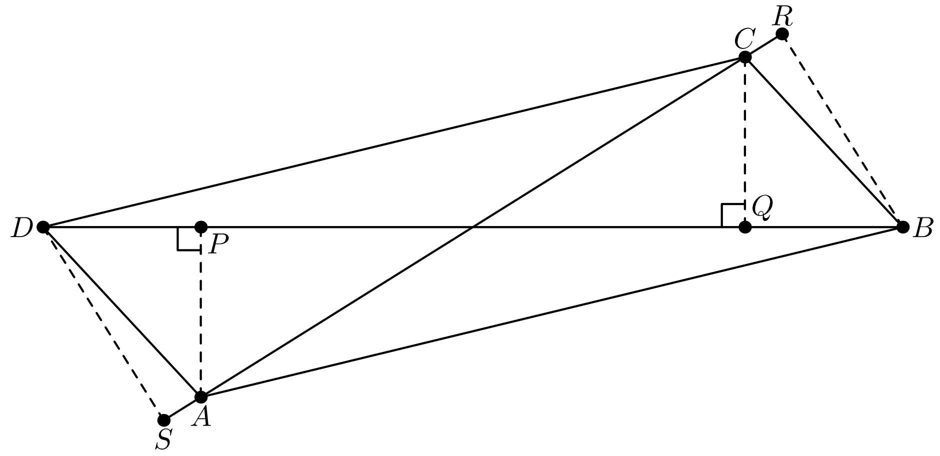
24 Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.



Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where $m, n,$ and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 81 **(B)** 89 **(C)** 97 **(D)** 105 **(E)** 113

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25 Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and $30,$ inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx.$ The possible values of m lie in an interval of length $\frac{a}{b},$ where a and b are relatively prime positive integers. What is $a + b?$

(A) 31 **(B)** 47 **(C)** 62 **(D)** 72 **(E)** 85
